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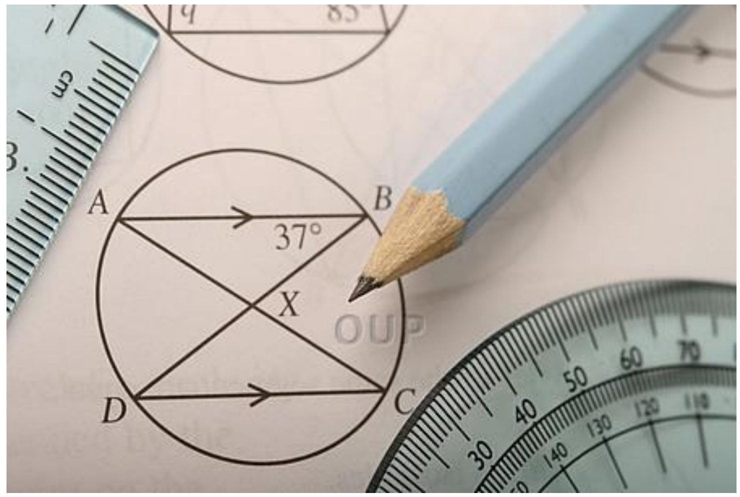
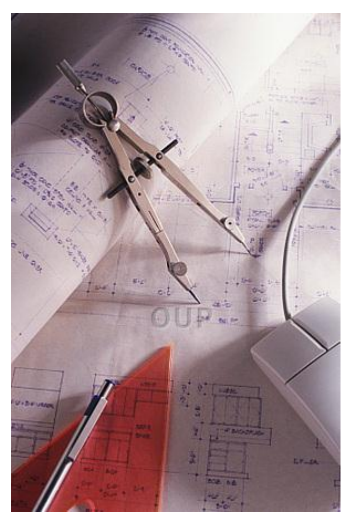
Trying a New 'Angle': Teaching Geometry through Stories

Sneha Titus



From the moment an infant moves his/her hands and legs, he/she begins to experience space. As we grow up, we explore and acknowledge our three-dimensional world and in order to negotiate our way successfully through it, we must develop a vocabulary of spatial concepts and understand the properties of these concepts. Shape and position are the key concepts that we use almost spontaneously every day - why then do we attempt to teach spatial concepts through definitions, diagrams and theorems? Geometry is of prime importance to the mathematics curriculum and if properly taught, can be very easily understood. But ask any middle school teacher for 'hard spots' in the curriculum and invariably, theorems and proofs in geometry appear in the list. What can we do differently so that this problem is mitigated?

Before we discuss probable solutions however, let us look at an important question – *Why teach geometry*?



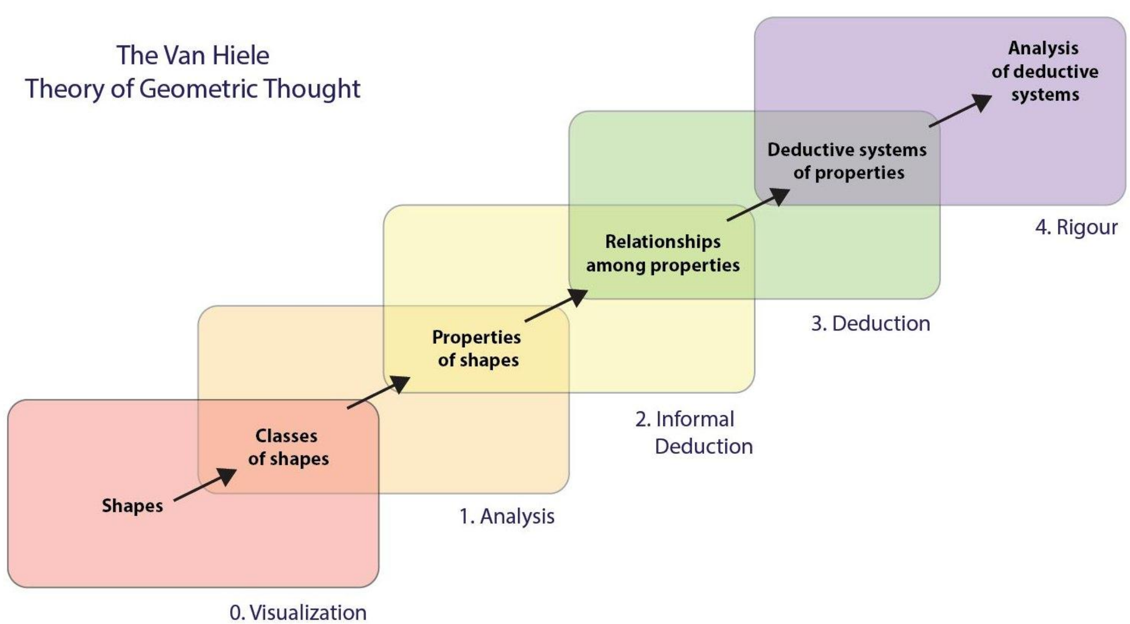
A standard definition of the term would be:

Geometry has been central to the historical development of mathematics itself. Through it were developed concepts such as abstraction, generalisation, deduction and proof. It still provides an avenue through which students can come to a deeper understanding of the nature of mathematics itself. (NZ, Ministry of Education, 2010)

According to *Clements* and *Sarama*, '*The ability to describe, use, and visualize the effects of composing, decomposing, embedding, and disembedding shapes* *is an important mathematical competence. It is relevant to geometry but also related to children's ability to compose and decompose numbers. Further, it underlies knowledge and skill with art, architecture, and the sciences. Thus, it (along with geometric measurement) helps people solve a wide variety of problems, from geometric proofs to the design of a floor space*.' So geometry is significant not only in itself but also as a means and not just an end.

Interestingly, research has shown that while many teachers are keenly aware of this fact, this message does not seem to have percolated to their students. Many students believe that the study of geometry is to enable them to master a body of mathematical knowledge, presented in the form of theorems and their proofs. Clearly, the teacher has reason to resort to a variety of pedagogical tools that exercise mathematical skills while delivering geometry-specific content.

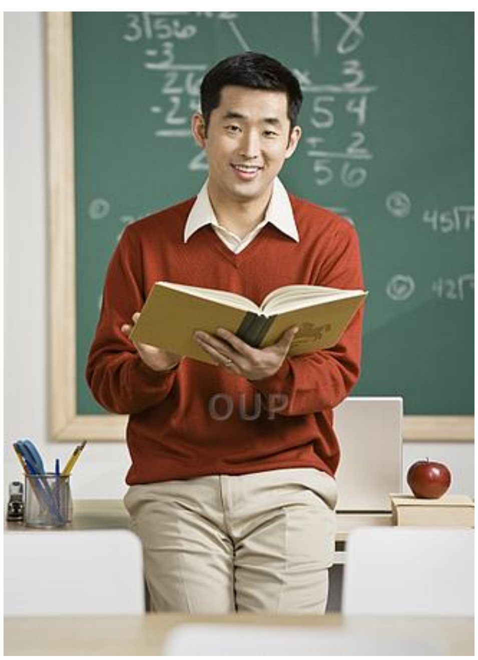
This content should enable students to visualize shapes, describe and analyze them, subsequently define them and find relationships between them. In addition, the content should enable them to develop and appreciate the need for logical arguments. Formal reasoning and rigour should come in only after this stage. What has been described above is nothing but the *Van Hieles*' levels of thought in geometry. All too often, students are hurled into the final level of reasoning, manipulation of geometric statements and rigour without going through the previous stages. '*Mathematisation*' of the child's thought processes, so highly valued by **NCF** (**National Curriculum Framework**) can only happen if the study of geometry takes the scenic route through the preliminary stages.



NCF 2005 also puts a high premium on mathematical communication, reasoning and problem solving. **NCTM** (**National Council of Teachers of Mathematics**) has the following to say: *In general, when researchers use the term* ***problem solving****, they are referring to mathematical tasks that have the potential to provide intellectual challenges that can enhance students' mathematical development. Such tasks—that is, problems—can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interests and curiosity (Hiebert & Wearne, 1993; Marcus & Fey, 2003; NCTM, 1991; Van de Walle, 2003).*

See the following *'worthwhile-problem'* criteria:

1. *The problem has important, useful mathematics embedded in it.*
2. *The problem requires higher-level thinking and problem solving.*
3. *The problem contributes to the conceptual development of students.*
4. *The problem creates an opportunity for the teacher to assess what their students are learning and where they are experiencing difficulties.*
5. *The problem can be approached by students in multiple ways using different solution strategies.*
6. *The problem has various solutions or allows different decisions or positions to be taken and defended.*
7. *The problem encourages student engagement and discourse.*
8. *The problem connects to other important mathematical ideas.*
9. *The problem promotes the skilful use of mathematics.*
10. *The problem provides an opportunity to practise important skills.*



Finally, stories have a long history of being used in many civilizations as pedagogical tools. A story or narrative helps students because of the personalization it brings to concepts and facts. Increasing involvement in the story helps students undertake problem solving as a quest which brings about a desirable end. In the story used in the lesson plan to teach the concept of congruency, the intention is for the student to arrive at the conditions of congruency rather than for the teacher to state and explain these conditions. The teacher plays the role of the narrator. The main protagonist is an adolescent just like the audience for whom the story is intended. Like all adolescents, she is not ready to accept statements unquestioningly. She wants to investigate, explore, deduce and solve problems.

Though the adventure of Scalene's hunt for a 'doppelganger' (*in fiction and folklore, a* ***doppelgänger*** *or* ***doppelganger*** *is a double of a living person. In contemporary vernacular, the word* ***doppelgänger*** *is often used in a more general sense to identify any person that physically or perhaps even behaviourally resembles another person.* Source: ***Wikipedia***) is eventually narrated in the story, the teacher is invited to halt the story at particular points and allow the students to interact with it. In other words, she can choose to set up the story in the form of a puzzle to be attempted by the students.

Polya describes the stages of problem solving as follows:

a. Understand the problem.
b. Isolate the data and conditions.
c. Find the connection between the data and the unknown (using an auxiliary problem if necessary) and arrive at a plan for the solution.
d. Carry out the plan – check each step, see if that step is correct. Can you prove if your step is correct?
e. Examine the solution obtained - Can you check the result? Can you check the argument? Can you derive the result differently? Can you use the result, or the method for some other problem?

Each candidate in the story presents a new problem and it is to be understood that students may not pass through these stages of problem solving in a linear fashion but will, naturally, go back and forth in their attempt to find a solution. However, the solution when they arrive at it will be completely their own. Scalene's solution is by intent given in a linear fashion and once students arrive at their solution, they can clarify their thinking by reading Scalene's solution of the problem. The questions in the lesson plan are intended to help the teacher diagnose the student's understanding of the story and their ability to visualize concepts described in the narration.

The final aim of the story is to develop most of the conditions for two triangles to be congruent. The last few questions in the lesson plan will enable the students to summarize the learning from the story and in the process of doing so, enunciate the required conditions. Happy reading!

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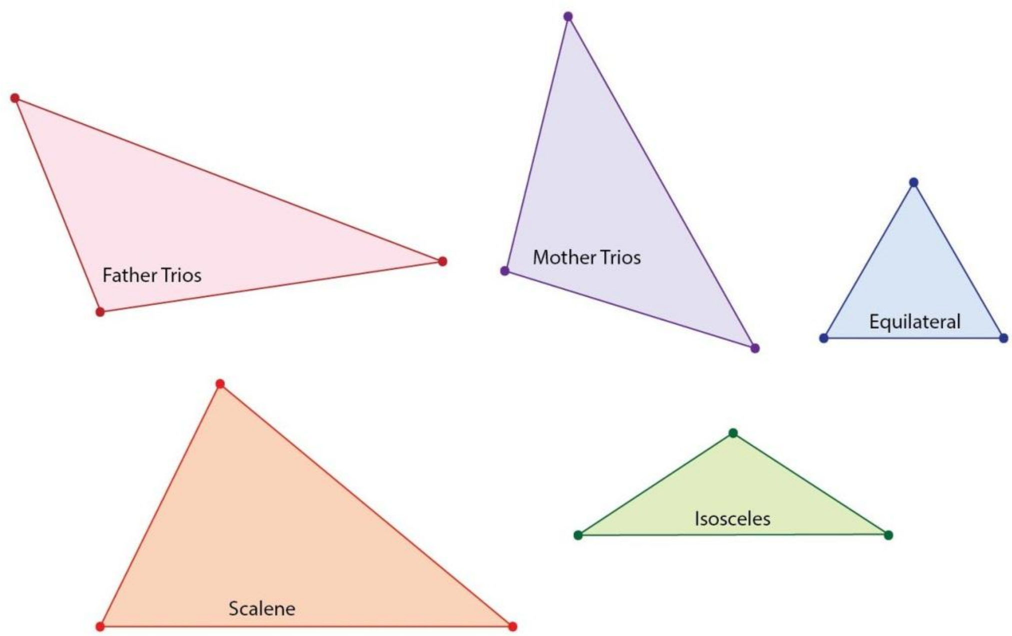
Lesson Plan on Teaching Geometry through Stories

Sneha Titus

Read the following story in the classroom, with the figures exhibited on a chart/drawn on the blackboard/whiteboard. Then give the worksheet to the students to solve, initiating a class discussion on the topic of congruence.

On the Doppelgänger Trail

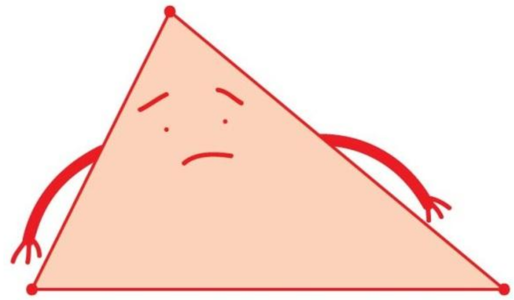
A small and interesting family called Trios lived in the land of Triangles. Father and Mother Trios had three children - Scalene, Isosceles and Equilateral. All five of them had a strong family resemblance as you can see in the picture below:



But then, each of them also looked quite different and to no one did this difference make such a difference as to the oldest child Scalene! You see, Scalene had reached adolescence and was quite preoccupied with looks. She complained bitterly to her mother, “I’m all angles! Isosceles and Equilateral look so complete and balanced and they are so… so… symmetrical! Each time people look at me, they think I look different! I stand out in a crowd and I feel lonely!”

Mother Trios was a sensible, practical lady and she didn’t make much of Scalene’s grumbling. But when the parents saw that Scalene was beginning to avoid the other Triangles and stay at home, instead of going out and meeting people, they decided it was time to tell her some facts of life. “You see, Scalene,” they began, “it’s not all about looking alike. It’s quite possible to stand out in a crowd and yet be part of it.” “But I want to find someone like me,” whined Scalene.

That's when Mother Trios lost her cool and snapped, "Well, moping at home is no way to go about this! Do you know that in this town of Triangles, you will be able to find many exact replicas of yourself? It's called a *doppelgänger* - your own alter identity. Go out and look for one of them!"



Scalene brightened up considerably! Now this was incredible news! She wouldn't mind roaming the streets of Triangles till she found her… *doppelgänger*? (That was a strange term if there ever was one!) And from that day, Scalene was a girl on a mission! Before she started out, she examined herself very carefully, and this was what she noticed:

- She had three vertices - A, B and C.
- She had three sides - AB = 8cm, BC = 7.2 cm and CA = 5.9 cm.
- She had three angles - angle A = 60 degrees, angle B = 45 degrees and angle C = 75 degrees (approximately).

She couldn't wait to meet her *doppelgänger*, her replica who had all these characteristics, which she had thought unique to herself till then. Surely she wouldn't feel like such a loner then!

And so began Scalene's search… She was a systematic sort and knew that roaming the streets would not be as effective as putting the word out on the internet. So that is exactly what she did!

'Looking for an exact twin!' – her Facebook post said. '*Suitable reward for the right applicant. Must possess the following characteristics:*

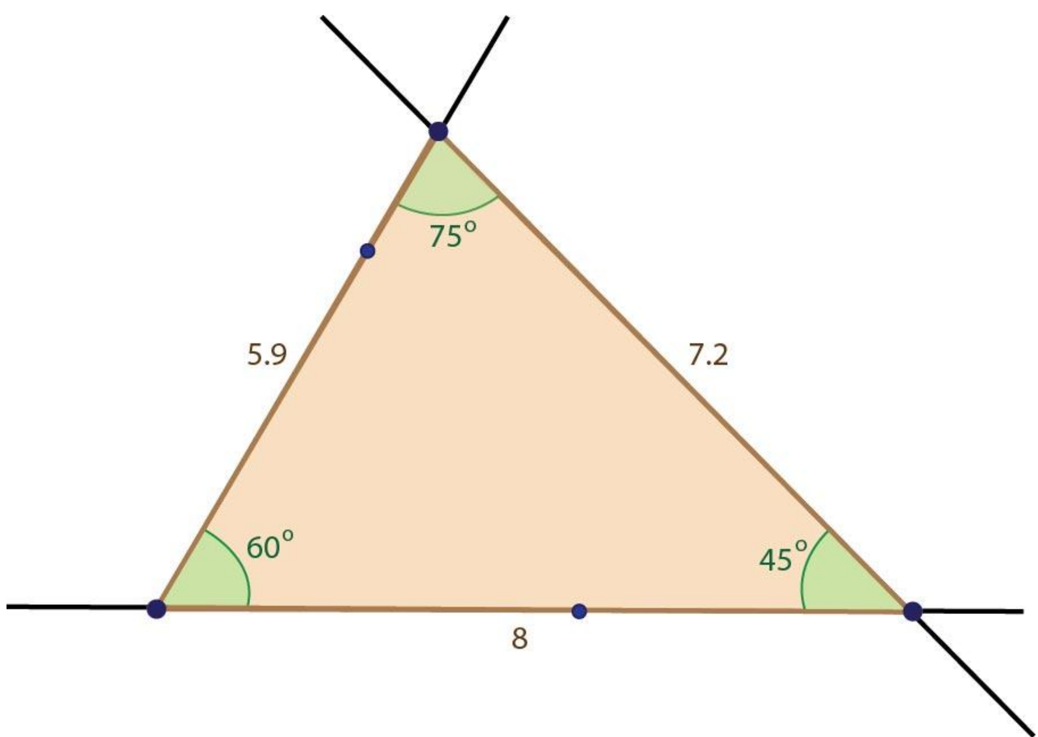
- *Three vertices*
- *Three sides of length - 8 cm, 5.9 cm and 7.2 cm.*
- *Three angles which are 60 degrees, 45 degrees and 75 degrees approximately.'*

Of course the promise of the reward immediately brought in responses to the post! Scalene went through them cautiously. The first announced that he had three vertices and two sides of length 5.9 cm and 7.2 cm. Could this be her replica? Scalene thought very carefully about this. She noticed that there had been no mention of angles. So she brought out her scientific kit and with a ruler and compass, she marked out two circles: one of radius 5.9 cm and another with its centre on the circumference of the first circle and with radius 7.2 cm. Immediately she saw that if the third point moved around the second circle, she could get a variety of triangles which had two sides of length 5.9 cm and 7.2 cm. Clearly, this candidate had good reasons not to mention the length of the third side!

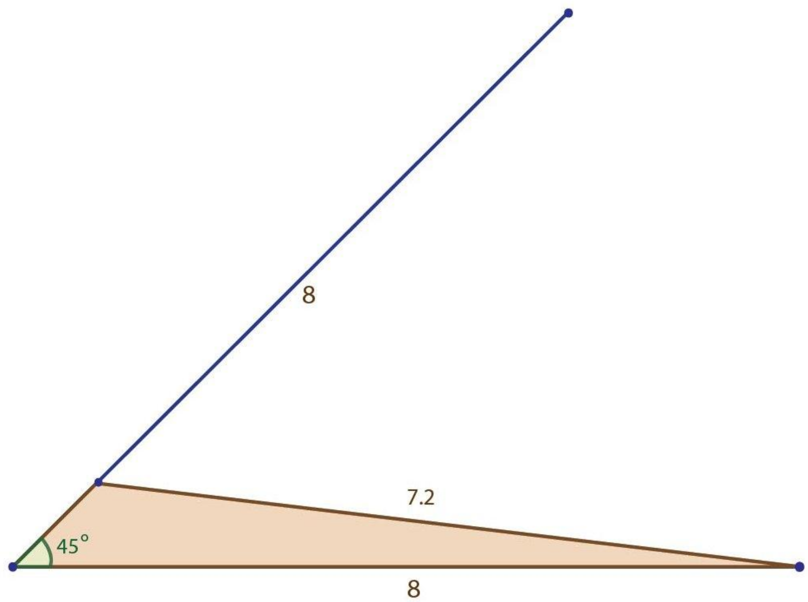
The second candidate announced that two of her angles were 60 degrees and 45 degrees. Out came the scientific kit again and Scalene quickly constructed a straight line and then 60 degrees and 45 degrees on either end of it. She noticed with excitement that the two rays marking these angles met at 75 degrees just as she did. But then she noticed that the length of the base line could vary and that these angles need not change. “This candidate is very similar to me,” she thought. “But we are not identical.”

Sieving through the applications, Scalene began to realize quite a few facts about others in the land of Triangles. All the applicants had at least two characteristics identical to her. But the lengths of two sides being the same was not sufficient to guarantee an exact match. Two angles being the same was enough to guarantee that the third angle was also the same; but even then the sides did not have to match necessarily.

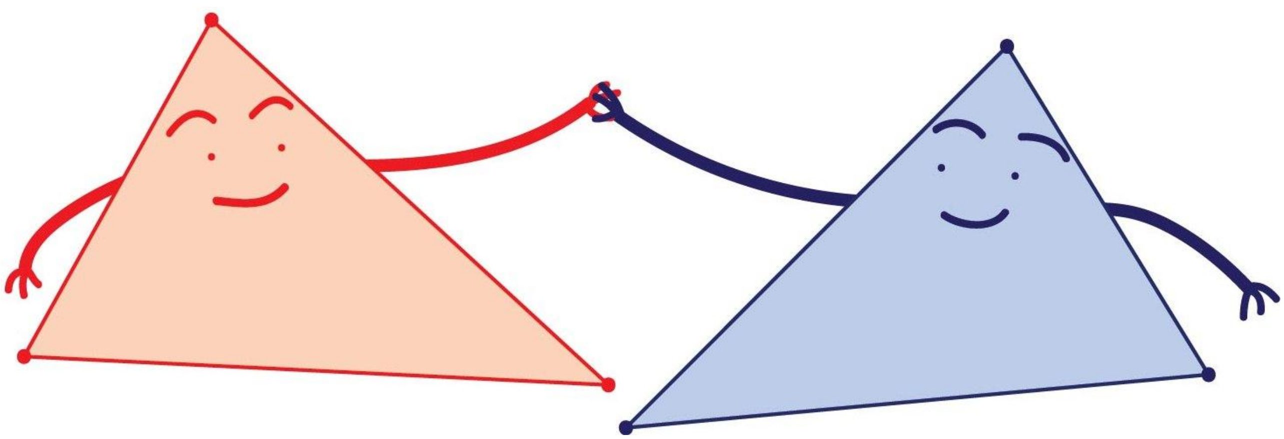
“I have to find a triangle whose 3 sides match mine,” she thought disconsolately. “Or if two angles match, then at least one side has to be the same.” So far no one had applied with these criteria. But Scalene knew that her mother would not have lied to her. So she discarded all the applications which specified only two characteristics and even those which had all three angles matching. Finally she came across an application which said that the lengths of two of the candidate’s sides were 7.2 cm and 8 cm, with one angle equal to 45 degrees. Excitedly, she drew the following figure:



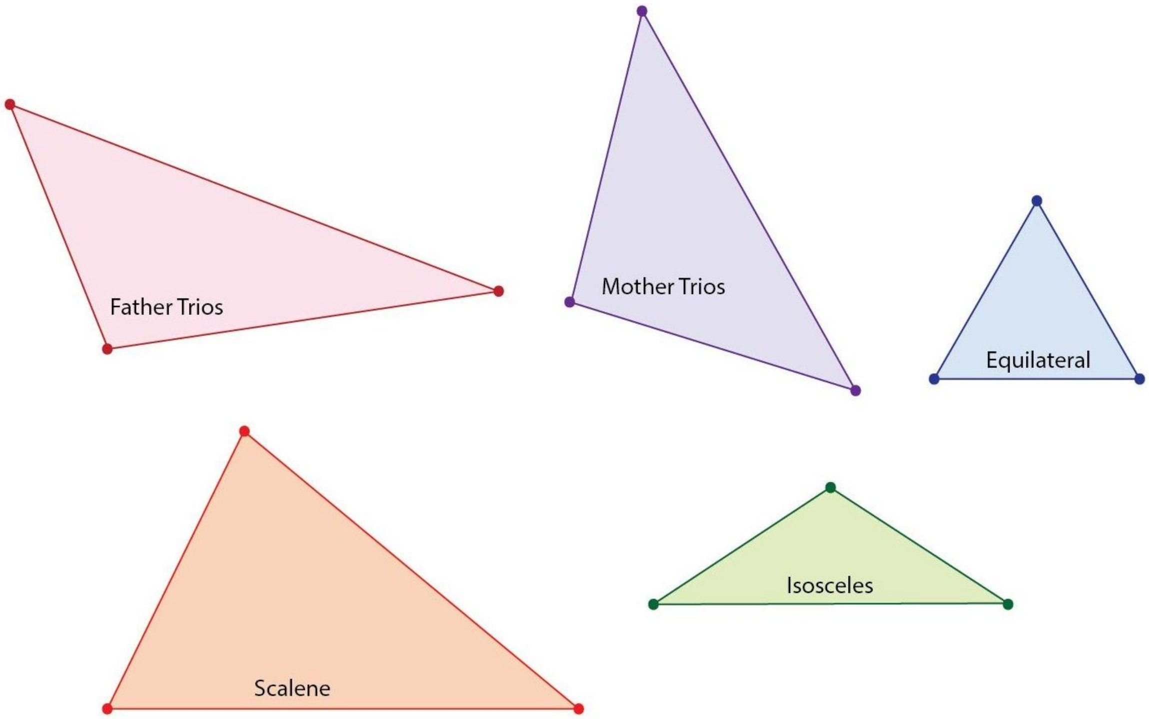
Surely she had found her perfect match! But then she re-read the application - “***An*** angle equal to 45 degrees?” Then it was possible that this candidate could also look like this?



Scalene was almost close to giving up, but thankfully, with the very next application, she realized that she had found her *doppelgänger*! This application said very clearly that the candidate had two sides of 8 cm and 5.9 cm, and that the angle included between them was 60 degrees. Once Scalene constructed this figure, there was only one way to complete the triangle! She couldn't wait to meet her *doppelgänger* - her mother was right and she was a loner no more! Now to rustle up a reward for her new friend!



Worksheet (Suitable even as a *GeoGebra* activity sheet)



1. What are the similarities between the members of the Trios family?

2. Use your ruler and protractor to find the differences between the members of the Trios family.

3. *She complained bitterly to her mother, "I'm all angles! Isosceles and Equilateral look so complete and balanced and they are so… so… symmetrical!"*

a) Is Scalene all angles?
b) Are Equilateral and Isosceles more symmetrical than Scalene? Justify your answer.

4. *She had three vertices - A, B and C.*

She had three sides - AB = 8cm, BC = 7.2 cm and CA = 5.9 cm.

She had three angles- angle A = 60 degrees, angle B = 45 degrees, and angle C = 75 degrees (approximately).

a) Construct and label triangle ABC using the given measurements. How many measurements did you use? State the ones you used and verify the remaining measurements.

5. *Clearly, this candidate had good reasons not to mention the length of the third side!*

a) Construct a triangle with two sides of length 5.9 cm and 7.2 cm. Measure the third side. Can it be 8 cm? Does it have to be 8 cm? (You can try this on *GeoGebra* too). If it is 8 cm, do the angles match?

6. *But then she noticed that the length of the base line could vary and that these angles need not change.*

a) Construct a triangle with two angles equal to 60 degrees and 45 degrees respectively. Join the two rays to form a triangle.
   (i) What is the measure of the third angle?
   (ii) What is the length of the base?
   (iii) What are the lengths of the two remaining sides?
   (iv) If the length of the base is 8 cm, will the triangle be an exact replica of Scalene?

7. *Finally she came across an application which said that the lengths of two of the candidate's sides were 7.2 cm and 8 cm, with one angle equal to 45 degrees.* How many triangles can be drawn which match the given requirements?

8. *This application said very clearly that this candidate had two sides of 8 cm and 5.9 cm, and that the angle included between them was 60 degrees.* How many triangles can be drawn which match the given requirements?

9. For candidate 1 to be the *doppelganger*, what additional information would have helped? Give all possible options.

10. For candidate 2 to be the *doppelganger*, what additional information would have helped? Give all possible options.

11. For candidate 3 to be the *doppelganger*, what specification had to be made?

12. Summarize the three conditions for Scalene to find a perfect *doppelgänger*.

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Words Section

Tessellate (noun)

Meaning

- To cover (a plane surface) by repeated use of one or more geometric shapes, without gaps or overlapping. In Mathematics, tessellations can be generalized to higher dimensions. *(Oxforddictionaries.com)*

Origin (and additional information) ~ The term's first known use was in late 17th century (as *tessellated*): it is derived from late Latin *tessellat-*, from the verb *tessellare* (meaning *to pave*), which in turn comes from *tessella*, a diminutive of *tessera*, referring to a small square tablet of wood or bone used as a token in ancient Greece and Rome.

Tessellations were used by the *Sumerians* (about 4000 BC) in building wall decorations formed by patterns of clay tiles. In 1619, *Johannes Kepler* made one of the first documented studies of tessellations when he wrote about regular and semi-regular tessellations, which are coverings of a plane with regular polygons, in his ***Harmonices Mundi***. Some two hundred years later in 1891, the Russian crystallographer, *Yevgraf Fyodorov* studied the periodic tiling of the plane, and his work marked the unofficial beginning of the mathematical study of tessellations. Other prominent contributors to this field include *Shubnikov* and *Belov* (1951); and *Heinrich Heesch* and *Otto Kienzle* (1963).

A periodic tiling has a repetitive pattern. Some special kinds include regular tilings with regular polygonal tiles all of the same shape, and semi-regular tilings with regular tiles of more than one shape and with every corner identically arranged. The patterns formed by periodic tilings can be categorized into 17 wallpaper groups. An *aperiodic* or *non-periodic* tiling uses a small set of tile shapes that cannot form a repetitive pattern. In the geometry of higher dimensions, a *space filling* or *honeycomb* is also called a *tessellation of space*.

Words Section

A real physical tessellation is a tiling made of materials such as cemented ceramic squares or hexagons. Such tilings may be decorative patterns, or may have functions such as providing durable and water-resistant pavement, floor or wall coverings. Historically, tessellations were used in Ancient Rome and in Islamic art such as in the decorative tiling of the *Alhambra Palace*. In the twentieth century, the work of *M.C. Escher* often made use of tessellations for artistic effect. Tessellations are also sometimes employed for decorative effect in quilting.

In computer graphics, the term *tessellation* is used to describe the organization of information needed to render the appearance of the surfaces of realistic three-dimensional objects.

Usage ~

i. *One room had a red <u>tessellated</u> floor and the main reception room a geometric mosaic, partly restored.*
ii. *The nature of their decoration, whether by painted plaster on walls or ceilings, or by <u>tessellated</u> and mosaic floors, compares well with that from the countryside.*
iii. *About half of the design still survives in the temple ruins, set within a purple <u>tessellated</u> border.*

Derivative ~ *tessellated,* adjective